STABLE MOMENT MAPPINGS AND SINGULAR LAGRANGIAN FIBRATIONS

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Abstract

We study singular Lagrangian fibrations given by moment mappings using cohomological methods. We give a theorem for the stability of these foliations and construct a symplectic version of Mather’s stable mapping theorem.

0. Introduction

We study singular Lagrangian fibrations given by moment mappings. Our starting point is a theorem obtained in the sixties by Rüssmann and extended by Vey in the seventies [17, 20]. This result states that given $n$ Poisson commuting holomorphic function germs $F_1, \ldots, F_n : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}, 0)$ such that $F_i = p_i q_i + r_i, \ r_i \in \mathcal{M}^2$, there exist biholomorphic map germs $\varphi : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^{2n}, 0), \ \psi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi$ is symplectic and

$$F_i \circ \varphi = \psi(p_1 q_1, \ldots, p_n q_n), \ \psi = (\psi_1, \ldots, \psi_n).$$

Here $\mathcal{M}$ denotes the maximal ideal of the local ring $\mathcal{O}_{\mathbb{C}^{2n}, 0}$.

The Rüssmann–Vey theorem implies that the singular Lagrangian fibration induced by the moment mapping germ $F = (F_1, \ldots, F_n)$ is stable in a neighbourhood of the origin. Stability means that if we perturb slightly the map $F$ in the space of moment mapping germs, then up to a change of coordinates in a neighbourhood of the origin in $\mathbb{C}^{2n}$ which preserves the symplectic structure, we get the same mapping germs. This particular kind of stability will be called $M$-stability.

In this paper, we prove that a moment mapping germ is $M$-stable provided that it is infinitesimally $M$-stable and that this condition occurs when the mapping defines an infinitesimal Lagrangian versal deformation of its zero fibre. For mappings of corank one, singular fibration we will also prove that the infinitesimal Lagrangian versality is necessary. Therefore we get an equivalence between Lagrangian versal families of curves and stable integrable systems of corank one. This equivalence was stated by Colin de Verdière without proof [2, section 4].

The paper is divided as follows.

In section 1, we recall the construction of the space of infinitesimal Lagrangian deformations of a Lagrangian variety [17, 18] (see also [5]).

In section 2, we consider the analog construction for moment mappings.

Section 3 contains the results of the paper and their proofs.

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It was proved by Eliasson that the Rüssmann–Vey theorem holds in the $C^\infty$ category for elliptic Hamiltonians [4]. More generally, it would be interesting to investigate the real $C^\infty$ stability for moment mappings.

This paper reproduces half of the preprint [8] with minor modifications. The results concerning singular Lagrangian varieties are to be published in [5].

1. A short review on deformations of Lagrangian varieties

1.1. The complex of Lagrangian infinitesimal deformations

We consider the space $\mathbb{C}^{2n} = \{(q_1, \ldots, q_n, p_1, \ldots, p_n)\}$ endowed with the standard symplectic form $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$. We denote by $M$ a domain in $\mathbb{C}^{2n}$. The Poisson bracket $\{f, g\}$ of two holomorphic functions $f, g : M \to \mathbb{C}$ can be defined by the formula

$$\{f, g\} \omega^n = df \wedge dg \wedge \omega^{n-1}.$$

A Lagrangian submanifold of $\mathbb{C}^{2n}$ is an $n$-dimensional complex analytic manifold on which the symplectic form vanishes. A Lagrangian variety is a purely $n$-dimensional analytic variety $L$ such that the smooth part of $L$ is a Lagrangian manifold.

There is a natural notion of an (embedded) deformation of a Lagrangian variety. We consider only the case of complete intersections. Let $(L, 0) \subset (\mathbb{C}^{2n}, 0)$ be the germ of a Lagrangian complete intersection. A deformation of $(L, 0)$ is given by a map

$$\Phi = (\Phi_1, \ldots, \Phi_n) : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0), \quad (\lambda, q, p) \mapsto \Phi(\lambda, q, p),$$

such that the restrictions of the $\Phi_i$’s to $\lambda = 0$ generate the ideal of $(L, 0)$. The condition of being Lagrangian is equivalent to $\{\Phi_i, \Phi_j\} = 0$ which is also equivalent to the condition

$$d\Phi_i \wedge d\Phi_j \wedge \omega^{n-1} \wedge d\lambda_1 \ldots \wedge d\lambda_k = 0, \quad \lambda = (\lambda_1, \ldots, \lambda_k).$$

Denote by $\mathcal{O}_m$ the ring of germs at $0 \in \mathbb{C}^m$ of holomorphic functions and let $\mathcal{O}_{L, 0}$ the quotient ring of $\mathcal{O}_{2n}$ by the ideal $I$ of the Lagrange variety germ $(L, 0)$. As $(L, 0)$ is the germ of a complete intersection, we have an identification $\mathcal{I}^2 = \mathcal{O}_{L, 0}$ given by a choice of generators of the ideal $I$. The complex of Lagrangian infinitesimal deformations (defined in [19]) which, in general, has terms $C_{L, 0} = Hom(\bigwedge \mathcal{I}^2, \mathcal{O}_{L, 0})$ takes the following form:

$$C_{L, 0} : 0 \to \mathcal{O}_{L, 0} \to \bigwedge^1 \mathcal{O}_{L, 0} \to \bigwedge^2 \mathcal{O}_{L, 0} \to \cdots \to \bigwedge^n \mathcal{O}_{L, 0} \to 0.$$

We consider the particular case $n = 2$ and refer to [19] and to [18] for the general case.

We use the identifications

$$\bigwedge^1 \mathcal{O}_{L, 0} \approx \mathcal{O}_{L, 0}^2,$$

$$\bigwedge^2 \mathcal{O}_{L, 0} \approx \mathcal{O}_{L, 0}.$$
Since $\Phi$ is a Lagrangian deformation, there exists $a, b \in \mathcal{O}_{2n}$ such that

$$\{\Phi_1, \Phi_2\} = a\Phi_1 + b\Phi_2.$$  

We define the first differential to be

$$\delta : \mathcal{O}_{L,0} \to \mathcal{O}_{L,0}^2$$

$$h \mapsto (\{h, \Phi_1\}, \{h, \Phi_2\})$$

and the second differential by

$$\delta : \mathcal{O}_{L,0}^2 \to \mathcal{O}_{L,0}$$

$$(m_1, m_2) \mapsto \{m_1, \Phi_2\} + \{\Phi_1, m_2\} - a\Phi_1 - b\Phi_2.$$  

(In the definition of the differential, we abusively denoted a function in $\mathcal{O}_{2n}$ and its projection in the factor ring $\mathcal{O}_{L,0}$ by the same symbol.)

It is readily verified that the first cohomology group of the complex $C_{L,0}$ is equal to the first-order Lagrangian deformations of the Lagrangian variety germ $(L,0)$ modulo infinitesimally trivial deformations, where the coordinate changes have to be symplectic [19]. There is no difficulty in introducing a complex relative to a Lagrangian deformation $\Phi : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ that is, a complex with parameters [5]. We denote this complex by $C_\Phi$.

1.2. Pyramidal Lagrangian varieties

On a Lagrangian variety $L \subset \mathbb{C}^{2n}$, we define a stratification as follows. Denote by $V_x$, $x = (q, p) \in \mathbb{C}^{2n}$ the vector subspace of $T_x \mathbb{C}^{2n}$ generated by the hamilton vector fields of the functions belonging to the ideal of $L$ at $x$.

The strata $L_j \subset L$ are defined by $L_j = \{x \in L : \dim V_x = j\}$. We have that $L = \bigcup_{j=0}^n L_j$.

**Definition 1** [19] The Lagrangian variety $L$ is called pyramidal if, for any $k$, the variety $L_k$ is of dimension at most $k$.

**Theorem 1** [5] The cohomology $\mathcal{O}_{k+n}$-modules $H^k(C_G)$ associated to a Lagrangian deformation $G : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ of a pyramidal Lagrangian complete intersection are modules of finite type.

**Remark 1** The theorem holds for general pyramidal Lagrangian deformations (without the assumption of being a complete intersection).

2. Integrable deformations of moment mappings

2.1. Equivalence of moment mappings

By moment mapping $F = (F_1, \ldots, F_n) : M \to S$, $M \subset \mathbb{C}^{2n}$, $S \subset \mathbb{C}^n$ we mean a holomorphic map given $n$ Poisson commuting functions $F_1, \ldots, F_n$ (that is, for all $i, j \leq n$, $\{F_i, F_j\} = 0$).
Let $\Lambda \subset \mathbb{C}^k$ be a neighbourhood of the origin. An integrable deformation over the base $\Lambda$ of a moment mapping $F$ is a deformation $G = (G_1, \ldots, G_n) : \Lambda \times M \to S$ of $F$, i.e. $G(0, \cdot) = F$, having Poisson commuting components.

**Definition 2** A moment mapping $F'$ is called $M$-equivalent to $F$, if there exist a symplectic biholomorphic mapping $\varphi$ and a biholomorphic mapping $\psi$ such that the following diagram commutes:

![Diagram](image)

Like for the case of mappings [10–16], we have a notion of stability for $M$-equivalence: a moment mapping is called $M$-stable if it does not admit non-trivial one-parameter deformations (the corresponding notion for mappings is a sometimes called homotopic stability). Similar notions hold for germs of moment mappings.

There are two notions of equivalence for a moment mapping: Lagrangian equivalence when the map is viewed as an $n$-parameter deformation of its zero fibre and $M$-equivalence when it is viewed as a mapping. This is similar to the case of mapping germs

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \quad p \leq n,$$

for which there is a notion of contact equivalence and a notion of right–left equivalence. In the first case $f$ is viewed as a system of $p$ equations depending on a parameter $\varepsilon \in \mathbb{C}^p$, in the second case $f$ is viewed as a mapping germ.

### 2.2. Infinitesimal integrable deformations of moment mappings

For a given moment mapping germ $F = (F_1, \ldots, F_n) : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$, we denote by $\bar{F}$ the $n$-parameter Lagrangian deformation defined by

$$\bar{F} : (\mathbb{C}^n \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0), \quad (s_1, \ldots, s_n, q, p) \mapsto (F_1(q, p) - s_1, \ldots, F_n(q, p) - s_n).$$

Similarly a $k$-parameter integrable deformation $G : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ of the moment mapping $F$ defines a $k + n$ Lagrangian deformation of the Lagrangian variety $F^{-1}(0)$ that we denote by $\bar{G}$.

Now, consider the vector space $O_F = (O_{2n}/F^{-1}O_n)$. We define the terms of the complex of integrable deformations of moment maps by

$$K_F : 0 \to O_F \to O_F^n \to \bigwedge^2 O_{2n}^n \to \cdots \to \bigwedge^n O_M^n \to 0.$$

The differential of this complex is induced by that of the complex $C_{\bar{F}}$. Therefore we have an exact sequence of complexes of modules

$$0 \to A_{\bar{F}} \to C_{\bar{F}} \to K_{\bar{F}} \to 0,$$
where the complex $A_F$ is defined by

$$A_F : 0 \to F^{-1}O_n \to (F^{-1}O_n)^n \to 0$$

and its only differential is equal to zero.

The cohomology of the complex $K_F$ is an $F^{-1}O_n$-module defined by the rule $a(z) \cdot b(q, p) = a(F(q, p)) \cdot b(q, p)$. The differentials of the complex are linear with respect to this $O_n$-module structure since

$$F_i \{ F_j, \alpha \} = \{ F_j, \alpha F_i \}.$$ 

It is readily verified that the first cohomology group of the complex $K_F$ is equal to the first-order integrable deformations of the moment mapping $F$ modulo infinitesimally trivial deformations, where the coordinate changes have to be symplectic.

2.3. Finiteness of the deformation module

**Definition 3** A moment mapping germ $F : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ is called pyramidal if the Lagrangian variety germ $F^{-1}(0)$ (resp $G^{-1}(0)$) is itself pyramidal.

**Theorem 2** The cohomology $O_{k+n}$-module $H^1(K_G)$ associated to an integrable deformation germ $G : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ of a pyramidal moment mapping germ is a module of finite type.

**Proof.** The exact sequence

$$0 \to A_G \to C_G \to K_G \to 0$$

induces a long exact sequence in cohomology

$$\cdots \to H^p(A_G) \to H^p(C_G) \to H^p(K_G) \to \cdots$$

The $O_{k+n}$-modules $H^p(C_G)$ are of finite type (Theorem 2) and as

$$H^p(A_G) = \begin{cases} 
O_{k+n} & \text{for } p = 0, \\
O_n^{p} & \text{for } p = 1, \\
0 & \text{for } p > 1,
\end{cases}$$

is of finite type, we get that $H^p(K_G)$ is also of finite type.

3. Symplectic Mather’s theory

3.1. Main results

The infinitesimal generators of the Lagrangian deformation associated to a moment mapping germ $F : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ are by definition of the Lagrange complex, the cohomology classes

$$[(1, 0, \ldots, 0), \ldots, [(0, \ldots, 0, 1)] \in H^1(C_L, 0), (L, 0) = F^{-1}(0).$$

We denote by $V$ the vector space that they generate.
The following theorem is the infinitesimal symplectic version of Mather’s theorem ‘\(K\)-versality implies \(R–L\) stability’ [10–15].

**Theorem 3** The \(O_n\)-module \(H^1(K_F)\) associated to a pyramidal moment mapping germ \(F : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)\) is equal to zero provided that the \(C\)-vector space \(V\) is equal to \(H^1(\mathbb{C}_{L,0})\) where \((L, 0) = F^{-1}(0)\).

**Proof.** The exact sequence \(O_n\)-modules

\[
0 \to A_F \to C_F \to K_F \to 0
\]

induces a long exact sequence of cohomology modules, which obviously splits at the second term. We get the exact sequence

\[
0 \to H^0(K_F) \to H^1(A_F) \to H^1(C_F) \to H^1(K_F) \to 0. \tag{1}
\]

We assert that there is an exact sequence

\[
0 \to \mathcal{M}H^1(C_{\hat{F}}) \to H^1(C_{\hat{F}}) \to H^1(C_{L,0}), \tag{2}
\]

where \(\mathcal{M}\) is the maximal ideal of the local ring \(O_n\).

Denote by \(\Phi_k : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)\) the \(k\)-parameter Lagrangian deformation defined by

\[
(s_1, \ldots, s_k, q, p) \mapsto (F_1(q, p) - s_1, \ldots, F_k(q, p) - s_k, F_{k+1}, \ldots, F_n(q, p)),
\]

so that \(\Phi_0 = F, \Phi_n = \hat{F}\). The exact sequences of complexes given by the multiplication by \(s_k\)

\[
0 \to C_{\Phi_{k-1}} \xrightarrow{s_k} C_{\Phi_k} \to C_{\Phi_{k-1}} \to 0
\]

induce long exact sequences in cohomology:

\[
\cdots \to H^p(C_{\Phi_{k-1}}) \xrightarrow{s_k} H^p(C_{\Phi_k}) \to H^p(C_{\Phi_{k-1}}) \to \cdots.
\]

As the fibres of \(F\) are reduced, one easily sees that

\[
H^0(C_{\Phi_k}) = O_k, \quad H^0(C_{\Phi_{k-1}}) = O_{k-1}.
\]

Therefore, the exact sequence splits and we get injections

\[
i_k : H^1(C_{\Phi_{k-1}}) \xrightarrow{s_k} H^1(C_{\Phi_k}).
\]

This chain of injections gives an injection

\[
i : H^1(C_{\hat{F}}) \xrightarrow{\mathcal{M}H^1(C_{\hat{F}})} H^1(C_{\hat{F}}).
\]

This proves the exactness of the sequences (2).

The assumption \(V = H^1(C_{L,0})\) obviously implies that last arrow of this sequence is surjective. As \(H^1(C_{\hat{F}})\) is an \(O_n\)-module of finite type (Theorem 1), we get by Nakayama lemma that \(H^1(C_{\hat{F}})\) is generated by the constant mappings. This means that in the exact sequence (1),
the map

\[ H^1(A_F) \to H^1(C_{\tilde{f}}) \]

is surjective; therefore \( H^1(K_{\tilde{f}}) = 0 \). This proves the theorem.

**Remark 2** I do not know if the converse to Theorem 3 holds in general. Nevertheless, in case the Lagrangian variety \( L \) is (locally) analytically equivalent to the product of a curve \( \Gamma \subset \mathbb{C}^2 \) by a trivial factor (or equivalently if the corank of \( F \) equals one)

\[ (L, 0) \approx (\Gamma \times \mathbb{C}^{n-1}, 0) \]

then the converse statement holds (see also Example 2 below). Let us sketch the proof. First take coordinates \((q, p)\) so that \( F = (f(q_1, p_1), p_2, \ldots, p_n) \). Then, as one easily sees, there are isomorphisms

\[ H^1(C_{L,0}) \approx H^1(C_{\Gamma,0}), \quad H^1(C_{\tilde{f}}) = H^1(C_{\tilde{f}}), \]

where \( \tilde{f} \) is the \( n \)-parameter deformation of the curve \( \Gamma \) defined by

\[ \tilde{f} : (\mathbb{C}^n \times \mathbb{C}^2, 0) \to (\mathbb{C}, 0), \quad (s_1, \ldots, s_n, q_1, p_1) \to (f(q_1, p_1) - s_1). \]

As explained in [6], the \( \mathcal{O}_n \)-module \( H^1(C_{\tilde{f}}) \) is isomorphic to the relative Brieskorn lattice of \( \tilde{f} \). Therefore, it follows from results of Brieskorn and Greuel [1, 8] that \( H^1(C_{\tilde{f}}) \) is a free module of rank \( \dim H^1(C_{\Gamma,0}) \). Thus, we have the isomorphisms

\[ H^1(C_{\tilde{f}}) \approx H^1(C_{\tilde{f}}) \approx H^1(C_{L,0}) \otimes \mathcal{O}_n. \]

From the exact sequence (1) we deduce that the equality \( H^1(K_{\tilde{f}}) = 0 \) implies that \( H^1(C_{\tilde{f}}) \) is generated by the classes of the constant mappings. The previous sequence of isomorphisms implies in turn that \( V = H^1(C_{L,0}) \).

**Theorem 4** A pyramidal moment mapping germ \( F : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0) \) is \( M \)-stable provided that \( H^1(K_{\tilde{f}}) = 0 \).

**Example 1** Consider the moment mapping germ \( F = (F_1, \ldots, F_n) \) defined by \( F_i = p_i q_i \) and denote by \((L, 0)\) the zero fibre of \( F \). A straightforward computation shows that \( F \) satisfies the conditions of Theorem 3. Thus, the moment mapping germ \( F \) is \( M \)-stable (compare [17, 20]).

**Example 2** Consider the moment mapping germ \( F = (F_1, F_2) \) defined by

\[ F_1 = p_1^2 + q_1^3 + q_2 q_1, \quad F_2 = q_2. \]

Denote by

\[ \Gamma = \{(q_1, p_1) : p_1^2 + q_1^3 = 0\} \]

the symplectic reduction by \( \{q_2 = 0\} \) of the zero fibre \( L \) of \( F \). A straightforward computation shows that the vector spaces \( H^1(C_{L,0}) \) and \( H^1(C_{\Gamma,0}) \) are isomorphic. As \( \Gamma \) is quasi-homogeneous,
there is a canonical identification

\[ H^1(C_{1,0}) ≃ \mathcal{O}/Jf, \]

where \( Jf \) is the Jacobian ideal of \( p^2_1 + q^3_1 \) (see for example, [2] or [6]). Therefore the vector space \( H^1(C_L) \) is generated by \([1,0]\) and \([q_1,0]\). As \([q_1,0]\) is equal to \([0,1]\), Theorem 3 applies. Thus, the moment mapping germ \( F \) is \( M \)-stable. There is no difficulty to extend this result to an arbitrary Lagrangian versal deformation of a curve.

3.2. Proof of Theorem 4

**Lemma 1** A one parameter integrable deformation

\[ G : (\mathbb{C} \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0) \]

of \( F \) is \( M \)-trivial provided that there exist a map germ \( \psi : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) and a function germ \( h : (\mathbb{C} \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}, 0) \) solving the equation

\[ \psi(\cdot, G) + \{h, G\} = -\frac{\partial G}{\partial \tau}. \]

**Proof.** We search for a relative symplectomorphism germ

\[ \varphi : (\mathbb{C} \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^{2n}, 0) \]

and a map germ

\[ A : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \]

such that the following equalities hold:

\[ F = A_\tau \circ (G_\tau \circ \varphi_\tau), \quad (A_0, \varphi_0) = (Id_n, Id_{2n}). \quad (3) \]

Here \( Id_n \) denotes the identity mapping in \( \mathbb{C}^n \), \( A_\tau \) denotes \( A(\tau, \cdot) \), \( \varphi_\tau = \varphi(\tau, \cdot) \) and so on.

Denote by

\[ A'_\tau(y) : T_\tau \mathbb{C}^n \to T_\tau \mathbb{C}^n, \quad T_\tau \mathbb{C}^n \approx \mathbb{C}^n, \]

the derivative of (a representative of) \( A \) at \( y \). Next, remark that if \( y \) is close enough to the origin then \( A'_\tau(y) \) is an invertible linear map.

Differentiating the first equation of the system (3) with respect to \( \tau \) at \( \tau = t \), we get

\[ A'_\tau(G_\tau \circ \varphi_\tau) \left[ \frac{d}{d\tau} \mid_{\tau=t} (G_\tau \circ \varphi_\tau) + \left( \frac{d}{d\tau} \mid_{\tau=t} G_\tau \right) \circ \varphi_\tau \right] + \left( \frac{d}{d\tau} \mid_{\tau=t} A_\tau \right) \circ (G_\tau \circ \varphi_\tau) = 0. \quad (4) \]

Define the time-dependent vector field germ \( v_\tau \) and the map germ

\[ \psi : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \]

by the formulae

\[ v_\tau(\varphi_\tau(t, x)) = \frac{d}{d\tau} \mid_{\tau=t} \varphi_\tau(t, x), \quad A'_\tau(G_\tau \circ \varphi_\tau) \psi_\tau = \frac{d}{d\tau} \mid_{\tau=t} A_\tau. \]
Multiplying (4) on the right by $\varphi_t^{-1}$ and on the left by $(A_t')^{-1}(G_t, \varphi_t)$, we get

$$L_n G_t + \psi_t^o G_t + \partial_t G_t = 0. \quad (5)$$

Since $\varphi$ is a relative symplectomorphism, the time-dependent vector field $v_t$ is a Hamilton vector field. Thus, there exists a holomorphic function germ $h : (\mathbb{C} \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}, 0)$ such that

$$L_n G_t = \{ h, G_t \}.$$

Moreover, with our notation conventions, we have $\psi_t^o G_t = \psi(t, G(t, \cdot))$. Thus, the equation becomes

$$\psi(\cdot, G) + \{ h, G \} = -\frac{\partial G}{\partial t}.$$

This proves the lemma.

In cohomological terms, the lemma asserts that $G$ is $M$-induced from $F$ provided that the cohomology class $[\partial G/\partial t] \in H^1(K_G)$ is equal to zero. We assert that the following implication holds:

$$H^1(K_F) = 0 \Rightarrow H^1(K_G) = 0.$$

The proof of this assertion will conclude the proof of the theorem.

We have an exact sequence of complexes induced by multiplication by $t$:

$$0 \to K_G^t \to K_G^t \to K_F^t \to 0.$$

This short exact sequence induces a long exact sequence in cohomology

$$\cdots \to H^p(K_G^t) \to H^p(K_G^t) \to H^p(K_F^t) \to \cdots.$$

As the fibres of $F$ are reduced we have that

$$H^0(K_G^t) = H^0(K_F^t) = 0.$$

Therefore the exact sequence starts with

$$0 \to H^1(K_G^t) \to H^1(K_G^t) \to H^1(K_F^t) \to \cdots.$$

Thus, the vanishing of $H^1(K_F^t)$ implies that the quotient of the $O_{n+k+1}$-module $H^1(K_G^t)$ by the maximal ideal of $O_{n+k+1}$ vanishes. As $H^1(K_G^t)$ is a module of finite type (Theorem 2), the Nakayama lemma implies in turn that $H^1(K_G^t) = 0$. This proves the assertion and concludes the proof of the theorem.

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